A variation of Gronwall's lemma

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January 9, 2009

Abstract

We prove a variation of Gronwall's lemma.

The formulation and proof of the classical Gronwall's lemma can be found in [1]. We prove here a variation of this lemma, which we were not able to find in the literature. The main difference from usual versions of Gronwall's lemma is that $-\lambda$ is *negative*.

Lemma 1 *Let* $g : [0, \infty[\to \mathbb{R}$ *be a continuous function,* C *a real number and* λ *a* positive *real number. Assume that*

$$\forall u, t \quad 0 \leq u \leq t \quad g(t) - g(u) \leq \int_u^t -\lambda g(s) + C ds \tag{1}$$

Then

$$\forall t \geq 0 \quad g(t) \leq \frac{C}{\lambda} + \left[g(0) - \frac{C}{\lambda} \right]^+ e^{-\lambda t} \tag{2}$$

where $[\cdot]^+ = \max(0, \cdot)$.

Proof **Case 1 :** $C = 0$, $g(0) > 0$.

Define $h(t)$ by

$$\forall t \geq 0 \quad h(t) = g(0)e^{-\lambda t}$$

Remark that h is positive with $h(0) = g(0)$, and satisfies (1) where the inequality has been replaced by an equality

$$\forall u, t \quad 0 \leq u \leq t \quad h(t) - h(u) = -\int_u^t \lambda h(s) ds$$

Consider now the set $S = \{t \geq 0 \mid g(t) > h(t)\}$. If $S = \emptyset$ then the lemma holds true. Assume by contradiction that $S \neq \emptyset$. In this case, consider an element $a \in S$. One has by definition $g(a) > h(a)$. Since $g(0) = h(0)$, one also has $a > 0$. Consider now

$$m = \inf\{a' < a \mid \forall t \in]a', a[\quad g(t) > h(t)\}$$

By continuity of g and h and by the fact that $g(0) = h(0)$, one has $g(m) = h(m)$. One thus also has $m < a$ and

$$\forall t \in]m, a[\quad g(t) > h(t) \tag{3}$$

Consider now $\phi(t) = g(m) - \lambda \int_m^t g(s)ds$. Equation (1) implies that

$$\forall t \geq m \quad g(t) \leq \phi(t)$$

In order to compare $\phi(t)$ and $h(t)$ for $t \in]m, a[$, let us differentiate the ratio $\phi(t)/h(t)$.

$$\left(\frac{\phi}{h}\right)' = \frac{\phi' h - h' \phi}{h^2} = \frac{-\lambda g h + \lambda h \phi}{h^2} = \frac{\lambda h(\phi - g)}{h^2} \geq 0$$

Thus $\phi(t)/h(t)$ is increasing for $t \in]m, a[$. Since $\phi(m)/h(m) = 1$, one can conclude that

$$\forall t \in]m, a[\quad \phi(t) \geq h(t)$$

which implies, by definition of ϕ and h, that

$$\forall t \in]m, a[\quad \int_m^t g(s)ds \leq \int_m^t h(s)ds \tag{4}$$

Choose now a $t_0 \in]m, a[$. Then one has by (3)

$$\int_m^{t_0} g(s)ds > \int_m^{t_0} h(s)ds$$

which clearly contradicts (4).

Case 2 : $C = 0$, $g(0) \leq 0$

Consider the set $S = \{t \geq 0 \mid g(t) > 0\}$. If $S = \emptyset$ then the lemma holds true. Assume by contradiction that $S \neq \emptyset$. In this case, consider an element $a \in S$. One has by definition $g(a) > 0$. Since $g(0) \leq 0$, one also has $a > 0$. Consider now

$$m = \inf\{a' < a \mid \forall t \in]a', a[\quad g(t) > 0\}$$

By continuity of g and by the fact that $g(0) \leq 0$, one has $g(m) = 0$. One thus also has $m < a$ and

$$\forall t \in]m, a[\quad g(t) > 0 \tag{5}$$

Choose now a $t_0 \in]m, a[$. Equation (1) implies that

$$g(t_0) \leq -\lambda \int_m^{t_0} g(s)ds \leq 0$$

which clearly contradicts (5).

Case 3 : $C \neq 0$

Define $\hat{g} = g - C/\lambda$. One has

$$\forall u, t \quad 0 \leq u \leq t \quad \hat{g}(t) - \hat{g}(u) = g(t) - g(u) \leq \int_u^t -\lambda g(s) + C ds = -\int_u^t \lambda \hat{g}(s)ds$$

Thus $\hat{g}$ satisfies the conditions of Case 1 or Case 2, and as a consequence

$$\forall t \geq 0 \quad \hat{g}(t) \leq [\hat{g}(0)]^+ e^{-\lambda t}$$

The conclusion of the lemma follows by replacing $\hat{g}$ by $g - C/\lambda$ in the above equation. □

Acknowledgments

The author would like to thank N. Tabareau and J.-J. Slotine for their helpful comments and V. Valmorin for having pointed out an error in an earlier version of the manuscript.

References

[1] I. Gikhman, A. Skorokhod. *Introduction to the theory of random processes*. WB Saunders Company, Philadelphia, 1969.